TOPIC P3: LET ME COUNT THE WAYS ...

SPOTLIGHT: ROLLING DICE AND YAHTZEE

Dice are one of the oldest randomization devices known to man. Egyptian tombs, dated from 2000 BC, were found containing dice and there have some evidence of dice in archaeological excavations dating back to 6000 BC. It is interesting to note that dice appeared to be invented independently by many ancient cultures across the world. In ancient times, the result of a die throw was not just considered luck, but determined by gods. So casting dice was often used as a way of making decisions such as choosing rulers or dividing inheritances. The Roman goddess, Fortuna, daughter of Zeus was believed to determine the outcome of a throw.

In the 19th and 20th centuries, standard six-sided dice became a basic component of many commercial board games that were developed. One of the most current popular games is Yahtzee that is played with five dice. The Hasbro game company (http://www.hasbro.com) presents the history of the game. Yahtzee was invented by a wealthy Canadian couple to play aboard their yacht. This "yacht" game was popular among the couple's friends, who wanted copies of the game for themselves. The couple approached Mr. Edwin Lowe, who made a fortune selling bingo games, about marketing the game. Mr. Lowe's initial attempts to sell the game of Yahtzee by placing ads were not successful. Lowe thought that the game had to be played to be appreciated and he hosted a number of Yahtzee parties and the game became very successful. The Milton Bradley company acquired the E. S. Lowe Company and Yahtzee in 1973 and currently more than 50 million games are sold annually.

PREVIEW

In this topic, we consider a situation where probabilities are easy to compute. Suppose we write down all of the outcomes of our random experiment in such a way so that all of the outcomes are equally likely. Then we can compute a probability of interest by counting the number of outcomes in the sample space and counting the number of outcomes in the event of interest. We review some basic rules helpful in counting outcomes.

In this topic, your learning objectives are to:

- Understand and apply the multiplication counting rule.
- Understand the notion of an arrangement and be able to compute the number of arrangements of distinct objects in a given application.
- Understand the use of a combinations rule when objects are selected without regard for order.
- Understand which counting rule is appropriate in a given application.

NCTM Standards

✓ In Grades 6-8, all students should compute probabilities for simple compound events, using such methods as organized lists, and tree diagrams,

✓In Grades 9-12, all students should understand the concepts of sample space and probability distribution and construct sample spaces and distributions in simple cases

EQUALLY LIKELY OUTCOMES

Assume we can write the sample space in such a way that the outcomes are equally likely. Then, applying the classical interpretation, the probability of each outcome will be

$$P(outcome) = \frac{1}{number\ of\ outcomes}.$$

If we are interested in the probability of some event, then the probability is given by

$$P(event) = \frac{number\ of\ outcomes\ in\ event}{total\ number\ of\ outcomes}.$$

This simple formula should be used with caution. To illustrate the use (and misuse) of this formula, suppose you have a box containing five balls of which three are red, one is blue, and one is white. You select three balls without replacement from the box – what is the probability that you choose all red balls?

Let's consider two representations of the sample space of this experiment.

Sample space 1: Suppose we don't distinguish between balls of the same color and don't care about the order in which the balls are selected. Then if we let R, B, W denote choosing a red, blue, and white ball respectively, then there are four possible outcomes:

$$S1 = \{\{R, R, R\}, \{R, R, B\}, \{R, R, W\}, \{R, B, W\}).$$

If these outcomes in S1 are assumed equally likely, then the probability of choosing all red balls is

$$\text{Prob(all reds)} = 1/4.$$

Sample space 2: Suppose instead that we distinguish the balls of the same color, so the balls in the box are denoted by R1, R2, R3, B, W. Then we can write down ten possible outcomes

$$S2 = \{\{R1, R2, R3\}, \{R1, R2, B\}, \{R1, R2, W\}, \{R1, R3, B\}, \{R1, R3, W\}, \{R2, R3, B\}, \{R2, R3, W\}, \{R1, B, W\}, \{R2, B, W\}, \{R3, B, W\}\}.$$

If we assume these outcomes are equally likely, then the probability of choosing all reds is

$$\text{Prob(all reds)} = 1/10.$$

If we compare our answers, we see an obvious problem since we get two different answers for the probability of choosing all reds. What is going on? The problem is that the outcomes in the first sample space S1 *are not* equally likely. In particular, the chance of choosing three reds (R, R, R) is smaller than the chance of choosing a red, blue and white (R, B, W) -- there is only one way of selecting three reds, but there are three ways of selecting exactly one red. On the other hand, the outcomes in sample space S2 are equally likely since we were careful to distinguish the five balls in the box, and it is reasonable that any three of the five balls has the same chance of being selected.

From this example, we have learned a couple of things. First, when we write down a sample space, we should think carefully about the assumption that outcomes are equally likely. Second, when we have an experiment with duplicate items (like three red balls), it may be preferable to distinguish the items when we write down the sample space and compute probabilities.

PRACTICE: EQUALLY LIKELY OUTCOMES

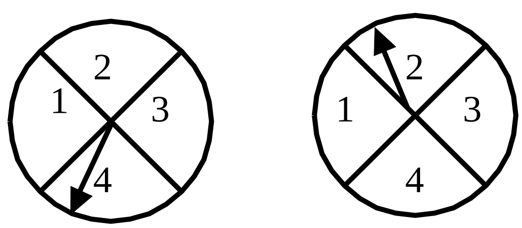
For each of the following experiments, are the outcomes in the given sample space equally likely? If the outcomes are not equally likely, explain why.

1. You will record the weather in your city next Monday. The sample space is S = {sunny, cloudy, rain, snow}.

2. You randomly choose a number from the group of digits {1, 2, 3, 4, 5, 6, 7} and record if the digit is even or odd. The sample space is S = {even, odd}.

3. Three people Ann, Bob, and Jacob are randomly put in a line. Suppose you record Ann's position in the line and the sample space is S={front of line, middle, back of line}.

Consider an experiment using two spinners:



SPINNER 1 SPINNER 2

4. Suppose you spin Spinner 1 and the sample space is S = {1, 2, 3, 4}.

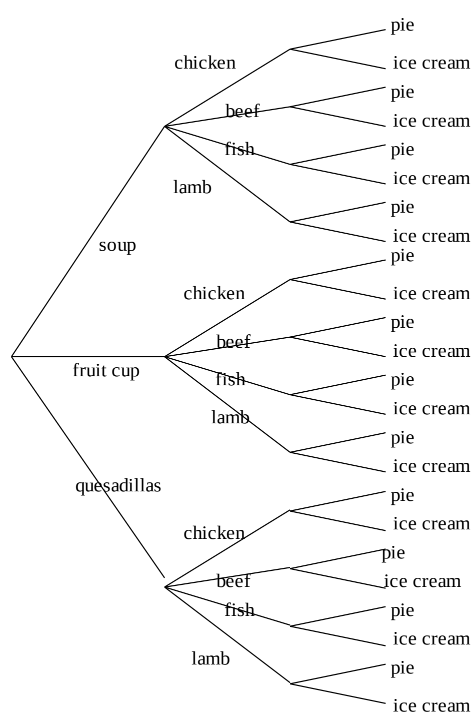
5. Suppose you spin both spinners and record the sum. The sample space is S = {2, 3, 4, 5, 6, 7, 8}.

THE MULTIPLICATION RULE

To apply the equally-likely recipe for computing probabilities, we need some methods for counting the number of outcomes in the sample space and the number of outcomes in the event. Here we illustrate a basic counting rule called the multiplication rule.

Suppose you are dining at your favorite restaurant. Your dinner consists of an appetizer, an entrée, and a dessert. You can either choose soup, fruit cup, or quesadillas for your appetizer, you have the choice of chicken, beef, fish, or lamb for your entrée, and you can have either pie or ice cream for your dessert.

We first use a tree diagram to write down all of your possible dinners. (The first set of branches shows the appetizers, the next set of branches the entrées, and the last set of branches the desserts.)



Note that there are 3 possible appetizers, 4 possible entrées, and 2 possible desserts. For each appetizer, there are 4 possible entrées, and so there are 3 x 4 = 12 possible choices of appetizer and entrée. Using similar reasoning, for each combination of appetizer and entrée, there are 2 possible desserts, and so the total number of complete dinners would be

Number of dinners = 3 x 4 x 2 = 24.

The above dining example illustrates a general counting rule that we call the multiplication rule.

MULTIPLICATION RULE: Suppose you are doing a task that consists of *k* steps. You can do the first step in n_1 ways, the second step in n_2 ways, the third step in n_3 ways, and so on. Then the number of ways of completing the task, which we will denote by *n*, is the product of the different ways of doing the *k* steps, or

$$n = n_1 \times n_2 \times \cdots \times n_k .$$

PRACTICE: THE MULTIPLICATION RULE

1. Suppose you are taking a one-way trip from Toledo to Buffalo to Rochester to New York City. You have two ways of driving from Toledo to Buffalo, three ways of driving from Buffalo to Rochester, and two ways of driving from Rochester to New York City. How many possible routes can you take on your trip?

2. Suppose you flip a penny, flip a quarter, and roll a six-sided die – you observe the side of the penny, the side of the quarter, and the roll on the die. How many possible outcomes are there?

3. Suppose you are ordering a large pizza and you see that the possible toppings are pepperoni, mushroom, extra cheese, peppers, ground beef, and ham. Some possible orders are pepperoni and extra cheese, ham, mushroom, and ground beef, everything (all six toppings), and plain (no toppings).

Suppose the waiter takes this order by asking you the following questions:

Q1. Do you want pepperoni?

Q2. Do you want mushrooms?

Q3. Do you want extra cheese?

Q4. Do you want peppers?

Q5. Do you want ground beef?

Q6. Do you want ham?

(a) How many possible answers are there to question Q1?

(b) How many possible answers are there to questions Q1 and Q2?

(c) How many possible answers are there to all six questions?

(d) How many possible ways can you order your pizza?

PERMUTATIONS

Suppose you load six songs, Song A, Song B, Song C, Song D, Song E, and Song F in your MP3 player. The songs are played in a random order and you listen to the first three songs. How many different selections of three songs can you hear?

In this example, we are assuming that the order that the songs are played is important. So hearing the selections

Song A, Song B, Song C

in that order will be considered different from hearing the selections in the sequence

Song C, Song B, Song A.

We call an outcome such as this a *permutation* or *arrangement* of 3 out of the 6 songs.

We can represent possible permutations by a set of three blanks, where we place songs in the blanks.

________	________	________
1^{st} Song	2^{nd} Song	3^{rd} Song

We find the number of permutations as follows:

1. First, we know that 6 possible songs can be played first. We place this number in the first blank above.

____6____	________	________
1^{st} Song	2^{nd} Song	3^{rd} Song

2. If we place a particular song, say Song A, in the first slot, there are 5 possible songs in the second position. We put this number in the second blank.

____6____	___5_____	________
1^{st} Song	2^{nd} Song	3^{rd} Song

By use of the multiplication rule, there are 6 x 5 = 30 ways of placing two songs in the first two slots.

3. Continuing in the same way, we see that there are 4 ways of putting a song in the 3^{rd} slot and completing the list of three songs.

____6____	____5____	___4_____
1^{st} Song	2^{nd} Song	3^{rd} Song

Again using the multiplication rule, we see that the number of possible permutations of six songs in the three positions are

$$6 \times 5 \times 4 = 120.$$

We have illustrated a second basic counting rule:

PERMUTATIONS RULE: If we have n objects (all distinguishable), then the number of ways to arrange r of them, called the number of *permutations*, is

$$\#of\ \ permutations = {}_nP_r = n \times (n-1) \times \cdots (n-r+1).$$

In this example, n = 6 and r = 3, and ${}_nP_r = 120$. If three songs are played in your MP3 player, each of the 120 possible permutations will be equally likely to occur. So the probability of any single permutation, say

Song A, Song D, Song B

is equal to 1/120.

Suppose you listen to all six songs on your player. How many possible orders are there? In this case, we are interested in finding the number of ways of arranging the entire set of 6 objects. Here n = 6 and r = 6 and, applying our formula, the number of permutations is

$${}_6P_6 = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

We use the special symbol n!, pronounced "n factorial", to denote the product of the integers from 1 to n. So the number of ways of arranging n distinct objects is

$${}_nP_n = n! = n \times (n-1) \times (n-2) \times \cdots \times 1.$$

PRACTICE: PERMUTATIONS

Suppose you are interested in constructing a four-letter word from the letters a, b, c, d, e, f, g. Here we use "word" to denote an arrangement of letters – it is very possible that the arrangement is not a real word.

1. In the permutation formula, what are the values of n and r?

2. How many possible four-letter words can you make?

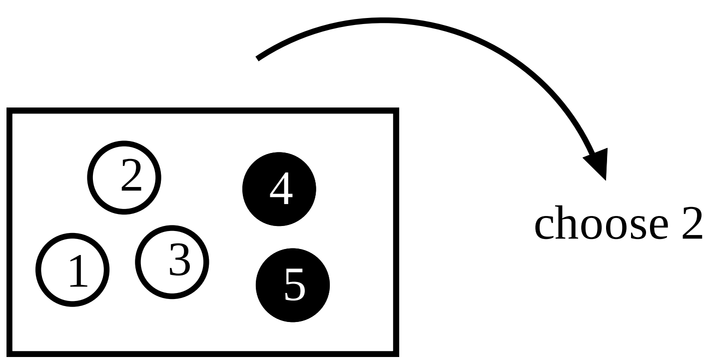
3. What is the probability a randomly arranged word is "bead"?

4. How many possible four-letter words begin with a vowel? (Construct this word in two steps: first choose a vowel for the first letter, and then choose the remaining three letters. Find the number of ways of performing each of the two steps and apply the multiplication rule.)

5. Suppose you wish to form a word using all seven letters. What are the values of n and r? Find the number of possible arrangements.

COMBINATIONS

Suppose you have a box with five balls -- three are white and two are black. You first shake up the box and then you choose two balls out *without replacement*. (This means that once you take a ball out, you do not return it to the box before you take the second ball out.)

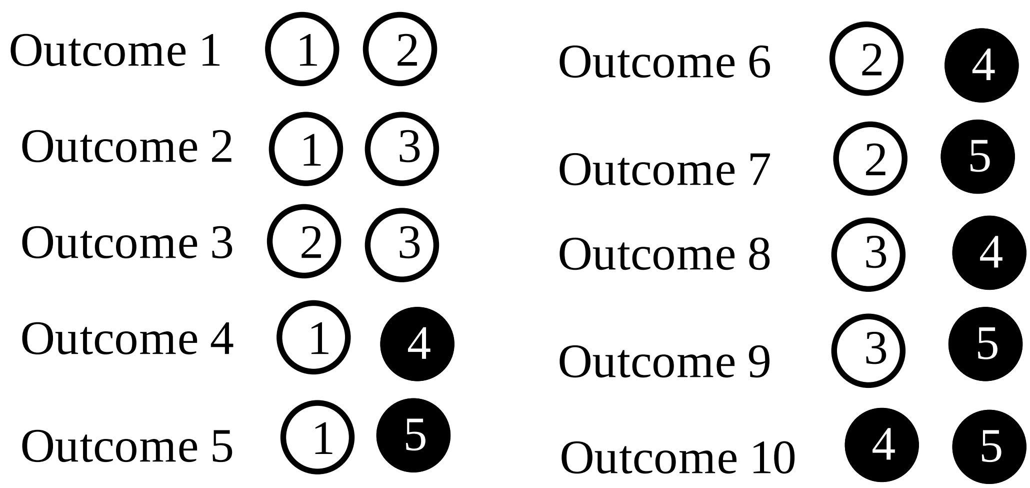


To make it easier to talk about outcomes, we have labeled the five balls from 1 to 5. Remember we are choosing two balls from the box and an outcome would be the numbers of the two balls that we select.

When we list possible outcomes, we should decide if it matters how we order the selection of balls. That is, if we choose ball 1 and then ball 2, is that different than choosing ball 2 and then ball 1?

We could say that order is important -- so choosing ball 1 then ball 2 is a different outcome from ball 2 then ball 1. But in this type of selection problem, it is common practice *not* to consider the order of the selection. Then all that matters is the collection or set of two balls that we select. In this case, we call the resulting outcome a *combination*.

When order doesn't matter, there are 10 possible pairs of balls that we can select. These outcomes or combinations are written below -- this list represents a sample space for this random experiment.



There is a simple formula for counting the number of outcomes in this situation. COMBINATIONS RULE: Suppose we have *n* objects and we wish to take a subset of size *r* from the group of objects without regards to order. Then the number of subsets or combinations, is given by the formula

$$number\ of\ combinations = {}_nC_r = \frac{n!}{r!(n-r)!}.$$

where $k!$ stands for k factorial $k! = k \times (k-1) \times (k-2) \times \cdots \times 2 \times 1$.

Let's try the formula in our example to see if it agrees with our number. In our setting, we have n = 5 balls and we are selecting a subset of size r = 2 from the box of balls. Using n = 5 and r = 2 in the formula, we get

$$_5C_2 = \frac{5!}{2!(5-2)!} = \frac{5\times4\times3\times2\times1}{[2\times1]\times[3\times2\times1]} = \frac{120}{12} = 10$$

that agrees with our earlier answer of 10 outcomes in the sample space.

PRACTICE: COMBINATIONS

PART A: Choosing balls from a box.

We continue with our example of choosing two balls from a box of five where three are white and two are black and there are $_5C_2 = 10$ possible combinations.

1. Assuming the outcomes are equally likely, what is the probability of each outcome?

2. Suppose we are interested in the probability that we choose exactly one white ball. To find this probability, we first want to count the number of outcomes that result in one white ball and one black ball.

(a) How many ways are there for choosing exactly one white ball?

(b) How many ways are there for choosing exactly one black ball?

(c) By the multiplication rule, how many ways are there for choosing one white ball *and* one black ball?

(d) Use the work in 1 and part 2 (c) to compute the probability of choosing one white ball.

3. Next consider the probability that we choose two balls of the same color. The first thing to realize is that "balls of the same color" means that either we are choosing two white balls or two black balls.

(a) How many ways can we choose 2 black balls?

(b) How many ways can we choose 2 whites?

(c) How many ways can we choose balls of the same color? (Add the numbers from parts (a) and (b).)

(d) Find the probability of choosing two balls of the same color.

PART B. Ordering a pizza

Let's return to our pizza example that we earlier discussed in this topic. We were interested in ordering a pizza and there were six possible toppings. We use the multiplication rule to compute the total number of possible pizzas that we could order.

How many toppings can there be in our pizza? Since there are six possible toppings, we could either have 0, 1, 2, 3, 4, 5, or 6 toppings on our pizza. Using combination formulas …

1. How many different pizzas can we order that have no toppings?
2. How many different one-topping pizzas can we order?
3. How many different two-topping pizzas can be order?
4. Find the number of possible three topping, four topping, five topping, and six topping pizzas. Record your answers (and the answers to 1, 2, and 3) in the below table.

Number of toppings	Number of ways
None	
One	
Two	
Three	
Four	
Five	
Six	
TOTAL	

5. To compute the total number of different pizzas, add the numbers in the table to get the total that we want. Compare your answer to the total number of pizzas that we computed in the earlier activity.

COMMENT: The above practice exercise demonstrated a general formula. Suppose we have a group of n objects and we are interested in the total number of subsets of this group. Then this total number is

$$2^n = {}_nC_0 + {}_nC_1 + \cdots + {}_nC_n.$$

The formula 2^n is found by noticing there are two possibilities for each object – either the object is in the subset or it is not – and then applying the multiplication rule. The right hand side of the equation is derived by first counting the number of subsets of size 0, of size 1, of size 2, and so on, and then adding all of these subset numbers to get the total number.

ARRANGEMENTS OF NON-DISTINCT OBJECTS

First let's use a simple example to review the two basic counting rules that we have discussed. Suppose you are making up silly words from the letters "a", "b", "c", "d", "e", "f", like

bacedf, decabf, eabcfd

How many silly words can I make up? Here we have n = 6 objects, and so the number of possible permutations is

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

To illustrate the second counting rule, suppose I have six letters "a", "b", "c", "d", "e", "f", and I am going to choose three of the letters to construct a three-letter word. I can't choose the same letter twice and the order in which I choose the letters is not important. In this case, we are interested in the number of combinations -- applying our combination rule with n = 6 and k = 3, the number of ways of choosing three letters from six is equal to

$${}_6C_3 = \frac{6!}{3!\,3!} = 20.$$

Let's consider a different arrangement problem. Suppose we randomly arrange the four triangles and five squares as shown below.

△ △ △ △ □ □ □ □ □

What is the chance that the first and last locations are occupied by triangles? This is an arrangement problem with one difference -- the objects are not all distinct -- we can't distinguish the four triangles or the five squares. So we can't use our earlier permutation rule that assumes the objects are distinguishable.

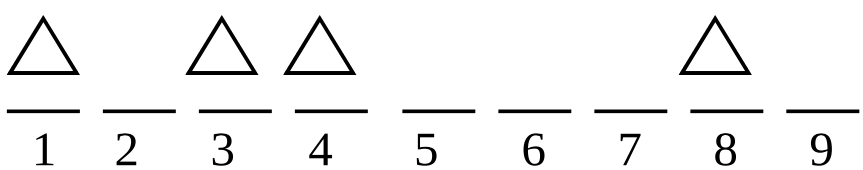
How can we count the number of possible arrangements? It turns out that a combination formula is useful here. (Surprising, but true.)

To think about possible arrangements, suppose we write down a list of nine slots and an arrangement is constructed by placing the triangles and the squares in the nine slots. It is helpful to label the slots with the numbers 1 through 9.

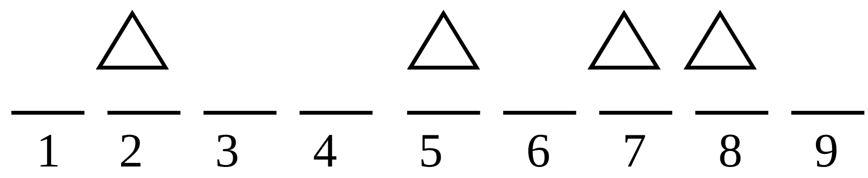
___	___	___	___	___	___	___	___	___
1	2	3	4	5	6	7	8	9

We construct an arrangement in two steps. First, we place the four triangles in four slots, and then we place the squares in the remaining slots.

How many ways can we put the triangles in the slots? First note that we can specify a placement by the numbers of the slots that are used. For example, we could place the triangles in slots 1, 3, 4, and 8.



Or we could place the four triangles in slots 2, 5, 7, and 8.



We can specify an arrangement by choosing four locations from the slot locations {1, 2, 3, 4, 5, 6, 7, 8, 9}. How many ways can we do this? That's easy. We know that the number of ways of selecting four objects (here labels of locations) from a group of nine objects is

$$ {}_9C_4 = \frac{9!}{4!\,(9-4)!} = 126. $$

So there are 126 ways of choosing the four locations for the triangles. Once we have placed the triangles, we can finish the arrangement by putting in the squares. But there is only one way of doing this. For example, if we place triangles in slots 2, 5, 6, 7, then the squares must go in slots 1, 3, 4, 8, 9. So applying the multiplication rule, the number of ways of arranging four triangles and five squares is 126 x 1 = 126.

We have derived a new counting rule:

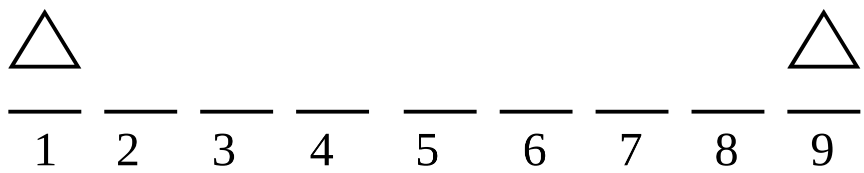
PERMUTATIONS RULE FOR NON-DISTINCT OBJECTS: The number of permutations of n *non-distinct* objects where r are of one type and n - r are of a second type is

$$ {}_nC_r = \frac{n!}{r!\,(n-r)!} $$

Recall our question that we wanted to answer: Suppose we randomly arrange four triangles and five squares. What is the chance that the first and last locations are occupied by triangles?

We have already shown that there are 126 ways of mixing up four triangles and five squares. Each possible arrangement is equally likely and has a chance of 1/126 of occurring.

To find our probability, we need to count the number of ways of arranging the triangles and squares so that the first and last positions are filled with triangles.



If we place triangles in slots 1 and 9 (and there is only one way of doing that), then we are free to arrange the remaining two triangles and five squares in slots 2, 3, 4, 5, 6, 7, 8, 9. By use of our new arrangements formula, the number of ways of doing this is

$$ {}_7C_2 = 21. $$

and so our probability the first and last slots are filled with triangles is equal to 21/126.

WHICH RULE?

We have described three important counting rules, the permutations rule, the combinations rule, and the permutations rule for non-distinct objects. How can you decide which rule to apply in a given problem? Here are some tips to help you find the right rule. The practice activity that follows will give you some experience in distinguishing between these rules.

1. Do We Care About Order?

If an outcome consists of a collection of objects, does the order that you list the objects matter? If order does matter, then a permutation rule may be appropriate. If the order of the objects doesn't matter, such as choosing a subset from a larger group, then a combination rule is probably more suitable.

2. Are the Objects Distinguishable?

We have two permutation formulas, one that applies when all of the objects are distinguishable, and the second where there are two types of objects and you can't distinguish between the objects of each type.

3. When In Doubt …

If the first two tips don't seem helpful, it may benefit to start writing down a few outcomes in the sample space. When you look at different outcomes, you should recognize if order is important and if the objects are distinguishable.

PRACTICE: THREE COUNTING RULES

For each of the following problems, state the appropriate counting rule (permutations for distinct objects, combinations, permutations for nondistinct objects) and use the rule to answer the problem.

1. Suppose you line up five coins, a penny, a nickel, a dime, a quarter, and a half-dollar, in a row. How many possible lineups can you have?

2. Suppose you line up 3 pennies and 2 nickels. How many possible lineups can you have?

3. Suppose you have five coins, a penny, a nickel, a dime, a quarter, and a half-dollar, in your pocket and you select three coins out. How many possible groups of coins can you have?

4. Suppose you flip a coin 10 times and record the sequence of heads and tails. If you flip exactly six heads, how many possible sequences are there? (One possible sequence is HHHTTTTHHH.)

PLAYING YAHTZEE

Yahtzee is a popular game played with five dice. The game is similar to the card game Poker – in both games, one is trying to achieve desirable patterns in the dice faces or cards, and some types of patterns are similar in the two games. In this section, we describe some of the dice patterns in the first roll in Yahtzee and consider the problem of determining the chances of several of the patterns.

Outcomes of one roll of five dice

When a player rolls five dice in the game Yahtzee, the most valuable result is when all of the five dice show the same number such as

2, 2, 2, 2, 2.

This is called a "Yahtzee" and the player scores 50 points with this pattern. A second valuable pattern is a "four-of-a-kind" where you observe one number appearing four times, such as

3, 4, 3, 3, 3.

The following table gives all of the possible patterns when you roll five dice in Yahtzee. When you play the game, some of these patterns are worth a particular number of points and these points are given in the right column.

Pattern	Sample of pattern	Point value
Yahtzee	4, 4, 4, 4, 4	50
Four of a kind	6, 6, 6, 4, 6	
Large straight	2, 6, 4, 5, 3	40
Small straight	4, 2, 1, 3, 2	30
Full house	5, 1, 1, 5, 1	25
Three of a kind	2, 2, 3, 4, 2	
Two pair	6, 3, 3, 6, 2	
One pair	4, 3, 4, 1, 5	
Nothing	1, 3, 2, 5, 6	
TOTAL		

Total number of outcomes

As in the case of two dice, it is useful to distinguish the five dice when we count outcomes. We can represent an outcome by placing a value of individual die rolls (1 through 6) in the six slots.

— — — — —

die 1 die 2 die 3 die 4 die 5

So two possible outcomes are

2, 3, 4, 5, 5 and 3, 2, 4, 5, 5

Each die has 6 possibilities and so, applying the multiplication rule, the total number of outcomes in the rolls of five dice is

$$6 \times 6 \times 6 \times 6 \times 6 = 7776.$$

Since all of the outcomes are equally likely, we assign a probability of 1/7776 to each outcome.

Probability of a Yahtzee

We can represent the Yahtzee roll as the outcome

x, x, x, x, x

where x denotes an arbitrary roll of one die. There are six possible choices for x, and so the number of possible Yahtzees is 6.

Since each outcome has probability 1/7776, the probability of a Yahtzee is

$$\text{Prob(Yahtzee)} = 6/7776.$$

Probability of four of a kind

In the pattern "four of a kind", we want to have one number appear four times and a second number appear once. In other words, we are interested in counting outcomes of the form

x, x, x, x, y

where the four x's and the single y can be in different orders.

To apply the multiplication rule, we think of writing down a possible "four of a kind" in three steps.

Step 1: We first choose the number for x (the number that appears four times).

Step 2: We next choose the number for the singleton y.

Step 3: We mix up the orders of the four x's and the one y.

For example, we choose the outcome 5, 5, 5, 3, 5 by (1) choosing 5 to be the number that appears four times, (2) choosing 3 as the number that appears once, and then arranging the digits 5, 5, 5, 5, 3 to get 5, 5, 5, 3, 5.

We next count the number of ways of doing each of the three steps.

Step 1: There are 6 ways of choosing x.

Step 2: Once x has been chosen, there are 5 ways of choosing the value for y.

Step 3: Last, once x and y have been selected, there are ${}_5C_4 = 5$ ways of mixing up the x's and y's.

To find the number of four-of-a-kinds, we use the multiplication rule using the number of ways of doing each of the three steps:

$$\text{Number of ways} = 6 \times 5 \times 5 = 150.$$

The corresponding probability of four-of-a-kind is

$$\text{Prob(four-of-a-kind)} = 150/7776.$$

PRACTICE: YAHTZEE

1. (Probability of a two pair.) We follow the same basic method as described above in counting the number of two-pairs. Represent an outcome by the sequence

$$x, x, y, y, z$$

where x and y denote the numbers that will each occur twice, and z is the number that appears one.

(a) How many ways are there for choosing the numbers x and y? (Note that since both x and y each appear twice, it is incorrect to say that the number of ways of choosing x and y is $6 \times 5 = 30$.)

(b) How many ways are there for choosing z, the identity of the number that appears once?

(c) The number of ways of mixing up the orders of x, x, y, y, z is

$$\frac{5!}{2!\,2!\,1!} = 30.$$

(This is a generalization of the earlier counting rule of arrangements of two different types.) Combine this result with the results of parts (a) and (b) to find the number of two-pairs.

(d) Find the probability of a two-pair.

2. (Probability of a large straight.) A large straight is observing five consecutive numbers in our roll. We write down a large straight in two steps:

Step 1: We write down the possible five numbers in the large straight.
Step 2: We arrange the numbers.

(a) How many ways can you do step 1? In other words, how many possible choices are there for the five numbers that make up the straight?

(b) How many ways can you arrange the five numbers in the straight?

(c) Combining the two steps, how many ways can you have a large straight?

(d) What is the probability of a large straight?

ACTIVITY – MOTHERS AND BABIES

DESCRIPTION: One day four babies were born at the local hospital. But for some reason, the nurses forgot to put identification bands on the babies, and decided (believe it or not) to give the babies back to the four mothers in some random fashion and hope for the best. How many babies will be correctly matched with the mothers? We will answer this question two ways, first using a simulation with cards, and then by listing all possible outcomes and using the classical notion of probability.

MATERIALS NEEDED: Sets of playing cards where one set contains eight cards: four red cards of different values, say seven, eight, Jack, and Queen, and four black cards of the same values.

METHOD 1. (Simulation) We will simulate this experiment using four red cards of different types (the moms) and four black cards of the same types.

- Put the four red cards down in a row.
- Mix up the four black cards and place them below the red cards.
- Count the number of matches.

Repeat this experiment 20 times – record your answers in the first table, and summarize your values in the second table.

TRIAL	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
# of matches																				

Number of matches	0	1	2	3	4
COUNT					
PROBABILITY					

METHOD 2. (Thinking of all possible outcomes) Suppose that the names of the four babies are Abby Albert, Bobby Brown, Cindy Crawford, and Darren Daulton. In your lab book, write down all ways of arranging or permuting the four first names (ABCD and ABDC are two possible arrangements). For each arrangement, count the number of matches. For example if babies ABDC are assigned to mom's ABCD, the number of matches is 2. Find the probabilities of the number of matches and put your answers in a table.

Number of matches	0	1	2	3	4
PROBABILITY					

ACTIVITY: SAMPLING FROM A BAG

DESCRIPTION: Pick up a lunch bag and 5 blocks – 2 blocks have one color and 3 blocks have a different color. Let's assume the colors are black and white (your colors may vary). Think of black as the darker color of the two colors you have. You put the 5 blocks in the bag, mix them up, and choose two out (without replacement) – how many blocks will be black? We will address this question first by doing a simulation, and then by enumerating all of the outcomes of the experiment.

MATERIALS NEEDED: A number of lunch bags, where each bag contains 5 blocks (these could be balls or dice), where 2 blocks have one color and 3 blocks have a different color.

METHOD 1 (Simulation)

Simulate this process 20 times.

- Put all the blocks in the bag and mix them up.
- Select two out without replacement.
- Record the number of blacks you see in your sample.

Put your answers in the table below.

TRIAL	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
# of black selected																				

Find the probability of choosing 0 black, 1 black, etc – put your answers in the table.

Number of blacks	0	1	2
PROBABILITY			

METHOD 2 (Listing outcomes)

Distinguish the blocks in the bag; if you have 2 black and 3 white blocks write them as {B1, B2, W1, W2, W3}. Suppose you keep track of the order of the blocks you select (so choosing B1, W1 is different from choosing W1, B1).

Write down all possible outcomes (selections of two blocks).

Find the probabilities of choosing 0 black, 1 black etc – put your answers in the table.

Number of blacks	0	1	2
PROBABILITY			

METHOD 3 (Using counting arguments)

1. If the five blocks are distinguishable, you select two blocks, and the order in which you select the blocks is important, how many possible outcomes are there?

2. (Continuation of 1.) Count the number of outcomes where you choose a black first and a white second.

3. Count the number of outcomes where you choose a white first and a black second.

4. Count the number of outcomes where you choose exactly one black ball. (Combine answers from parts 1 and 2.)

5. Find the probability of choosing exactly one black.

WRAP-UP

In this topic, we introduced several counting rules helpful for computing probabilities in the case where the outcomes in the sample space are equally likely. The basic rule is the *multiplication rule* that counts the number of outcomes when the experiment consists of several stages. A *permutations* rule is appropriate when one is arranging a set of distinct objects. In contrast, a *combinations* rule is used when one is selecting a subset of a larger group without replacement. We discussed a third permutations rule useful for counting the number of arrangements of nondistinct objects. To decide on the appropriate counting rule, one should carefully consider possible outcomes of the random experiment. One should ask if the order of selecting items is important and if the items selected are all distinguishable.

EXERCISES

1. **Constructing a Word**

Suppose you select three letters at random from {a, b, c, d, e, f} to form a word.

a. How many possible words are there?

b. What is the probability the word you choose is “fad”?

c. What is the probability the word you choose contains the letter “a”?

d. What is the chance that the first letter in the word is “a”?

e. What is the probability that the word contains the letters “d”, “e”, and “f”?

2. **Running a Race**

There are seven runners in a race – three runners are from Team A and four runners are from Team B.

a. Suppose you record which runners finish first, second, and third. Count the number of possible outcomes of this race.

b. If the runners all have the same ability, then each of the outcomes in (a) are equally likely. Find the probability that Team A runners finish first, second, and third.

c. Find the probability that the first runner across the finish line is from Team A.

3. **Rolling Dice**

Suppose you roll three fair dice.

a. How many possible outcomes are there?

b. Find the probability you roll three sixes.

c. Find the probability that all three dice show the same number.

d. Find the probability that the sum of the dice is equal to 10.

4. **Ordering Hash Browns**

When you order Waffle House’s world famous hash browns, you can order them scattered (on the grill), smothered (with onions), chunked (with ham), topped (with chili), diced (with tomatoes), and peppered (with peppers). How many ways can you order hash browns at Waffle House?

5. **Selecting Balls from a Box**

A box contains 5 balls -- 2 are white, 2 are black, and one is green. You choose two balls out of the box at random without replacement.

a. Write down all possible outcomes of this experiment. (Assume that the order in which you select the balls is important.)

b. Find the probability that you choose two white balls.

c. Find the probability you choose two balls of the same color.

d. Find the probability you choose a white ball second.

6. **Dividing into Teams**

Suppose that ten boys are randomly divided into two teams of equal size. Find the probability that the three tallest boys are on the same team.

7. **Choosing Numbers**

Suppose you choose three numbers from the set {1, 2, 3, 4, 5, 6, 7, 8} (without replacement).

a. How many possible choices can you make?

b. What is the probability you choose exactly two even numbers?

c. What is the probability the three numbers add up to 10?

8. **Choosing People**

Suppose you choose two people from three married couples.

a. How many selections can you make?

b. What is the probability the two people you choose are married to each other?

c. What is the probability that the two people are of the same gender?

9. **Football Plays**

Suppose a football team has five basic plays, and they will randomly choose a play on each down.

a. On three downs, find the probability that the team runs the same play on each down.

b. Find the probability the team runs three different plays on the three downs.

10. **Playing the Lottery**

In a lottery game, you make a random guess at the winning three-digit number (each digit can be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9). You win $200 if your guess matches the winning number, $20 if your guess matches in exactly two positions and $2 if your guess matches in exactly one position. Find the probabilities of winning $200, winning $20, and winning $2.

11. **Dining at a Restaurant**

Suppose you are dining at a Chinese restaurant with the menu given below. You decide to order a combination meal where you get to order one soup or appetizer, one entrée (seafood, beef, or poultry), and a side dish (either fried rice or noodles).

a. How many possible combination meals can you order?

b. If you are able to go to this restaurant every day, approximately how many years could you order different combination meals?

c. Suppose that you are allergic to seafood (this includes crab, shrimp, and scallops). How many different combination meals can you order?

d. Suppose your friend orders two different entrées completely at random. How many possible dinners can she order? What is the probability the two entrées chosen contain the same meat?

SOUP
HOT AND SOUR SOUP
WONTON SOUP
EGG DROP SOUP

APPETIZERS
EGG ROLL
BARBECUED SPARERIBS
FRIED CHICKEN STRIPS
BUTTERFLY SHRIMP
CRAB RANGOON

SEAFOOD
SHRIMP WITH GARLIC SAUCE
CURRY SHRIMP
KUNG PAO SCALLOPS
FLOWER SHRIMP
SHRIMP WITH PEA PODS

BEEF
KUNG PAO BEEF
HUNAN BEEF
SZECHUAN STYLE BEEF
ORANGE BEEF (HOT & SPICY)

POU LT R Y
KUNG PAO CHICKEN
HUNAN CHICKEN
CHICKEN WITH DOUBLE NUTS
CHICKEN WITH GARLIC SAUCE
CURRY CHICKEN

FRIED RICE
CHICKEN FRIED RICE
BEEF FRIED RICE
SHRIMP FRIED RICE
PORK FRIED RICE
THREE DELIGHT FRIED RICE
VEGETABLE FRIED RICE

NOODLES/RICE
PAN FRIED NOODLES
MOO SHU PANCAKE
CHOW MEIN NOODLES
STEAMED RICE .

12. **Ordering Pizza**

If you buy a pizza from Papa John’s, you can you order the following toppings: ham, bacon, pepperoni, Italian sausage, sausage, beef, anchovies, extra cheese, baby portabella mushrooms, onions, black olives, Roma tomatoes, green peppers, jalapeno peppers, banana peppers, pineapple, grilled chicken.

a. If you have the option of choosing two toppings, how many different two topping pizzas can you order?

b. Suppose you want your two toppings to be some meat and some peppers. How many two-topping pizzas are of this type?

c. If you order a “random” two-topping pizza, what is the chance that it will have peppers?

d. If you are able to order at most four toppings, how many different pizzas can you order?

13. **Mixed Letters**

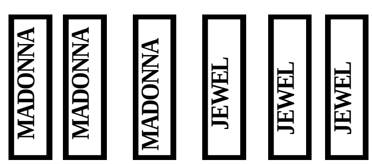
You randomly mix up the letters “s”, “t”, “a”, “t”, “s”.

a. Find the probability the arrangement spells the word “stats”.

b. Find the probability the arrangement starts and ends with “s”.

14. **Arranging CDs**

Suppose you have three Madonna cds and three Jewel cds sitting on a shelf as follows. (We assume that you can’t distinguish the cds of a given artist.)



The cds are knocked off of the shelf and you place them back on the shelf completely at random.

a. What is the probability that the mixed-up cds remain in the same order?

b. What is the probability that the first and last cds on the shelf are both Jewel music?

c. Which artist do you prefer, Madonna or Jewel?

d. What is the probability that the Jewel cds stay together on the shelf?

15. **Playing a Lottery Game**

The Minnesota State Lottery has a game called Daily 3. A three digit number is chosen randomly from the set {000, 001, …, 999} and you win by guessing correctly certain characteristics of this three digit number. The lottery website lists the following possible plays such as First Digit, Front Pair, etc. Find the probability of winning for each play.

First Digit *Pick one number. To win, match the first number drawn.*

Front Pair *Pick 2 numbers. To win, match the first 2 numbers drawn in exact order*

Straight *Pick 3 numbers. To win, match all 3 numbers drawn in exact order.*

3-Way Box *Pick 3 numbers, 2 that are the same. To win, match all three numbers drawn in any order.*

6-Way Box *Pick 3 different numbers. To win, match all 3 numbers drawn in any order.*

16. **Booking a Flight**

Suppose you are booking a flight to San Francisco on Orbitz. To save money, you agree to either leave Monday, Tuesday, or Wednesday, and return on either Friday, Saturday, or Sunday. Assume that Orbitz randomly assigns you a day to leave and randomly assigns you a day to return.

a. What is the probability you leave on Tuesday and return on Saturday?

b. What is the chance that your trip will be exactly three days long?

c. What is the most likely trip length in days?

d. Do you think that the assumptions about Orbitz are reasonable? Explain.

17. **Assigning Grades**

A math class of ten students takes an exam.

a. If the instructor decides to give exam grades of A to two randomly selected students, how many ways can this be done?

b. Of the remaining eight students, three will receive B's and the remaining will receive C's. How many ways can this be done?

c. If the instructor assigns at random, two A's, three B's and five C's to the ten students, how many ways can this be done?

d. Under this grading method, what is the probability that Jim (the best student in the class) gets an A?

18. **Choosing Officers**

A club consisting of 8 members has to choose three officers.

a. How many ways can this be done?

b. Suppose that the club needs to choose a president, a vice-president, and a treasurer. How many ways can this be done?

c. If the club consists of 4 men and 4 women and the officers are chosen at random, find the probability the three officers are all of the same gender.

d. Find the probability the president and the vice-president are different genders.

19. **Playing Yahtzee**

Find the number of ways and the corresponding probabilities of getting all of the following patterns in Yahtzee. Here are some hints for the different patterns.

Four of a kind: The pattern here is {x, x, x, x, y}, where x is the number that appears four times and y is the number that appears once.

Small straight: This roll will either include the numbers 1, 2, 3, 4, the numbers 2, 3, 4, 5, or the numbers 3, 4, 5, 6. If the numbers 1, 2, 3, 4 are the small straight, then the remaining number can not be 5 (otherwise it would be a large straight).

Full house: The pattern here is {x, x, x, y, y}, where x is the number that appears three times and y is the number that appears twice.

Three of a kind: The pattern here is {x, x, x, y, z}, where x is the number that appears three times, and y and z are the numbers that appear only once.

One pair: The pattern here is {x, x, w, y z}, where x is the number that appears two times, and w, y and z are the numbers that appear only once.

Nothing: This is the most difficult number to count directly. Once the number of each of the remaining patterns is found, then the number of "nothings" can be found by subtracting the total number of other patterns from the total number of rolls (7776).